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Model of a Joint Measurement of Different Spin Components

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A Stern–Gerlach type experiment is presented as an unsharp joint measurement of spin components. Using a simple realistic magnetic field, the interaction Hamiltonian reads $H = \mu(-\sigma_x x + \sigma_z z)$. In the framework of operational quantum physics the apparatus is represented by a Gaussian-shaped wave function. The probability reproducibility condition leads to the nearly best joint spin-x-z-observable formulated as a positive operator-valued measure. Observable and measurement are discussed within the quantum theory of measurement.

1. INTRODUCTION

The concern of this paper is the understanding of the joint spin observable, described first by Prugovečki (1977) and later by Busch and Schroeck (1989), and how it emerges from a realizable measurement (Martens and de Muynck, 1993). In accordance with operational quantum physics (Busch *et al.*, 1995b), the measurement can be regarded as the defining property of the measured observable. Using the quantum theory of measurement as formulated in Busch *et al.* (1996), the transition of the noncommuting spin properties toward commuting momentum properties is described explicitly by premeasurement, with and without reading. This way the two major interests, incommensurability and objectification, can be well separated. The premise of the treatment is the possibility of the objectification of commuting observables.

In the following section the measurement model is introduced. Section 3 is concerned with the positive operator-valued (POV) measure and the instrument or state transformation-valued (STV) measure. The characterization of the measurement is treated in Section 4; some final remarks are given in Section 5.

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2. THE MODEL

The Stern–Gerlach type experimental setup is sketched in Fig. 1. An oven emits atoms which possess a resulting magnetic moment, e.g., silver atoms. These atoms pass through a magnetic field and finally impinge on a photographic plate. The description of the experiment in Hilbert space is illustrated in the lower part of Fig. 1. The Hilbert space of the whole atom is the tensor product of the Hilbert space of the center-of-mass motion and that of the spin. Here the center of mass corresponds to the measuring apparatus. We assume the distribution of the center of mass initially to be Gaussian shaped, the density operator being $\rho^i_A = |\psi^i_A\rangle\langle\psi^i_A|$, with $\psi(x, z) = (1/\alpha)\exp[-\frac{1}{2}(x^2 - z^2)/(\Delta^2\hbar^2)]$, where α gives the normalization, and Δ is the width. Let the magnetic field of the Stern–Gerlach experiment be given by $B = B_0(-x, 0, z)$ The Hamiltonian then reads $H = -\mu_B\, B_0(-x\sigma_x + z\sigma_z)$, where μ_B is the Bohr magneton and σ_i are the Pauli matrices. As the pointer or readout observable the sharp momentum observable is taken, which is defined by the projection-valued measure $\chi \mapsto \int_X |p_x p_z\rangle\langle p_x p_z|\, dx\, dz$ on the photographic plate. It is called the impulsive measurement approximation (Busch and Schroeck, 1989; Kienzler, 1996) and maps the momentum space consistently on the position on the screen. Thus within a good approximation the probability measure for the momentum can be regarded as the distribution of the particles impinging on the screen. Figure 2, left, shows the planar symmetry, which is the result of a totally symmetric initial object state, and

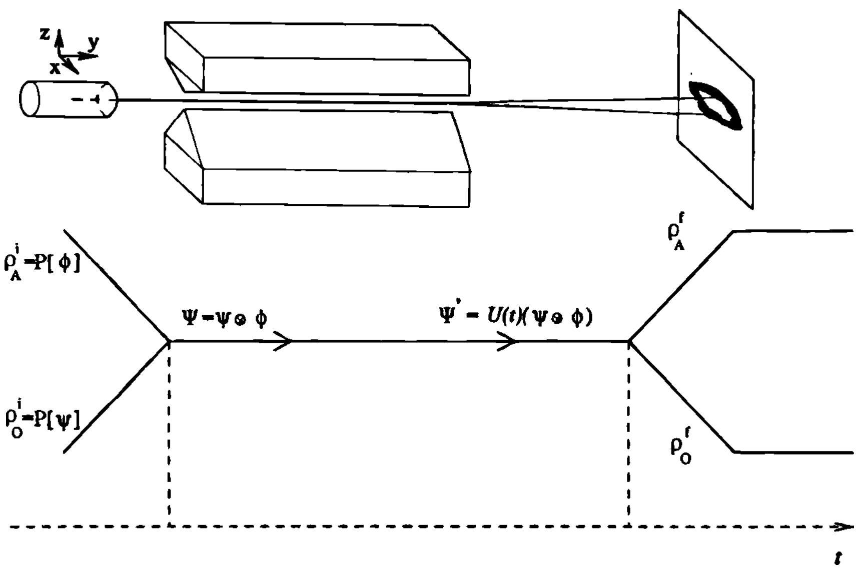

Fig. 1. The experimental setup and its description in Hilbert space.

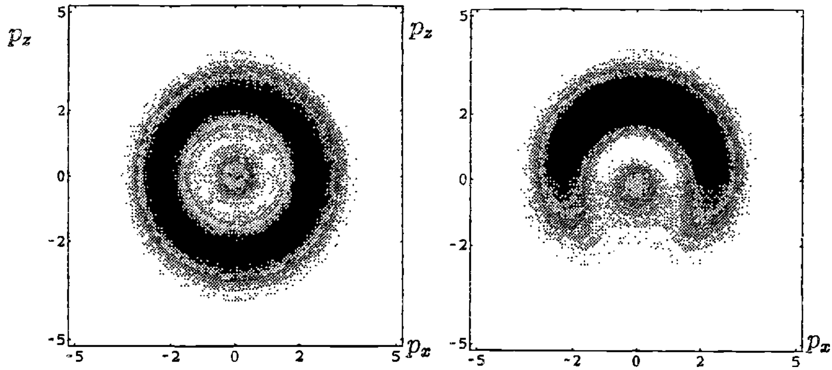

Fig. 2. On the left, the planar symmetry of the resulting picture of the screen with the initial state $(1/2)I$ and on the right with $|+_x\rangle\langle +_x|$ in units of the initial width of the wave packet [for details see Kienzler (1996)].

Fig. 2, right, shows the correlation of an initial spin-up state with the probability for the atoms to hit the upper half of the screen. To complete the measurement scheme one has to introduce a pointer function *f*, which maps the value space of the momentum observable on the value space of the observable, which is obtained through this measurement. The intuitive way, considering Fig. 2, is to map the upper left quarter of the screen to $++$, the upper right quarter to $-+$, and so on.

3. OBSERVABLE AND INSTRUMENT

Having a measurement scheme $\mathcal{M}$ consisting of the Hilbert space of the apparatus $\mathcal{H}_{\mathcal{A}}$, the unitary evolution $U = e^{-(i/\hbar)Ht}$, the pointer function f defined above, the initial state of the apparatus ρ_A^i, and the pointer observable $P_x \otimes P_z$, one can consider the probability reproducibility condition as the defining property of the observable E that is being measured (Busch *et al.*, 1996). It reads as follows (*tr* denotes the trace and X the set in which to measure), $\forall X\, \forall\, \rho_{Ob}$,

$$tr_{Ob}[\mathbf{E}(\mathbf{f}^{-1}(\mathbf{X}))\rho_{Ob}^i]$$
$$= tr_{\mathcal{H}}[U(\rho_A^i \otimes \rho_{Ob}^i)U^+ \int_X |p_x p_z\rangle\langle p_x p_z\, dx\, dz \otimes I] \qquad (1)$$

According to this the observable is a normalized mapping of the Borel sets $\{\pm, \pm\}$ into the positive operators on C^2. As is well known, the resulting effects as positive operators acting on $\underline{C}^2$ can be represented by $(\alpha, \underline{\mu}) \in R^4$, i.e., $E_{x,z}^{\pm,\pm} = (\alpha/2)(I + \underline{\mu}\,\underline{\sigma})$, with $|\mu| \leq 1$ and $\alpha \leq 2/(1 + |\mu|)$. Using the discretized version, it is easy to show that because of the planar symmetry the value of α is directly proportional to the size of the corresponding Borel sets. Furthermore, it is possible to combine the strength of the magnetic field B_0, the interaction time t_0, and the width of the initial wave packet of the apparatus into one interaction parameter $\xi = \mu B_0 t \Delta/\sqrt{\hbar}$. The resulting effects depend just on the absolute value of $\mu = |\mu| = \mu(\xi)$. This is shown in Fig. 3. The effects now read

$$E_{x,z}^{++} = \tfrac{1}{4}[I + \mu(\sigma_x + \sigma_z)], \qquad E_{x,z}^{+-} = \tfrac{1}{4}[I + \mu(\sigma_x - \sigma_z)] \qquad (2)$$

$$E_{x,z}^{--} = \tfrac{1}{4}[I + \mu(-\sigma_x - \sigma_z)], \qquad E_{x,z}^{-+} = \tfrac{1}{4}[I + \mu(-\sigma_x + \sigma_z)] \qquad (3)$$

Let us emphasize the problem of the size of the eigenvalues. It is easy to show that the eigenvalues do not exceed 1/2. Because of this there is no way to consider them as properties. Just the marginal effects corresponding to the reading on the half of the screen can be interpreted as still very weak properties, because even they cannot exceed $1/\sqrt{2}$, the limiting value of the theoretically best achieved joint spin observable (Busch, 1986).

Now it is shown that the realistic measurement defines a joint spin observable that is very close to the best possible joint observable. The question arises whether there is a possibility to ascribe this weak property to the object system. For this one has to regard the dynamics on the object level, i.e., the state transformation. The state transformation is represented using the Poincaré sphere, that is, the parameter representation of the effects with $\alpha = 1$.

If λ^i is the Poincaré vector of the initial state, the final state after the measurement is represented by $\lambda^f_{x/z} = \frac{1}{2}[1 + \mathscr{L}(\xi)]\lambda^i_{x/z}$ and $\lambda^f_y = \mathscr{L}(\xi)\lambda^i_y$,

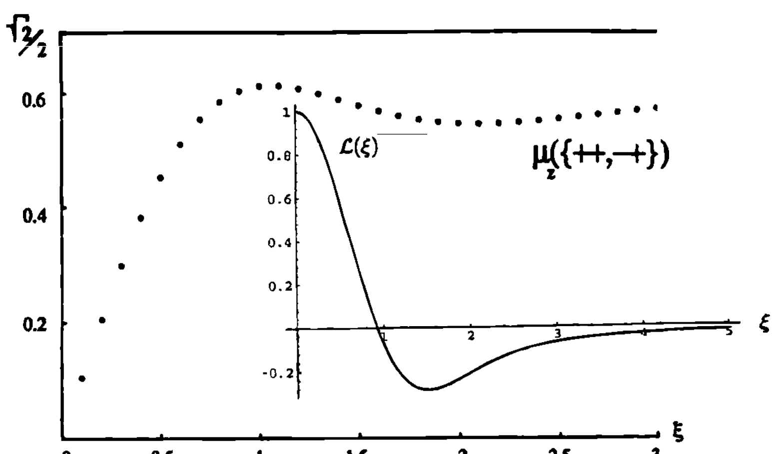

Fig. 3. The increase of the parameter μ of the nontrivial part of the effects as a function of the interaction parameter ξ. On the right the decrease of nontriviality of the state is shown.

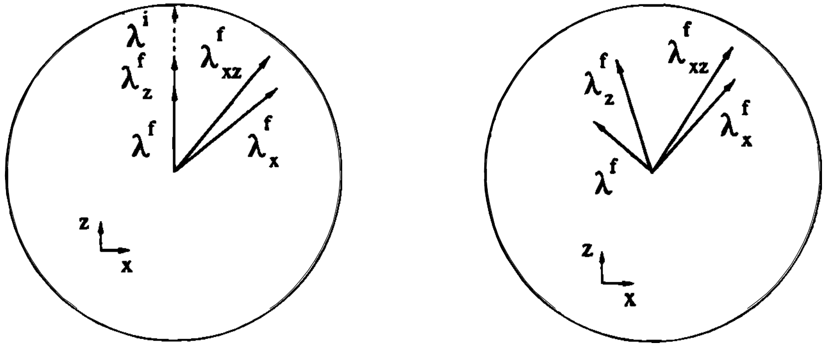

Fig. 4. The state transformation from the initial state λ^i to the final state without reading λ^f and respectively for *x, z,* and *x* and *z* reading for a spin-up state (left) and a totally symmetric initial state (1/2) *I* (right).

where $\mathcal{L}(\xi) = L_{1/2}^{1/2}$ is the generalized Laguerre polynomial (see Fig. 3). If the measurement with reading is considered, the state transformation explicitly depends on the Borel set of the reading scale. The state change by the instrument is represented in Fig. 4. The left side of the figure shows a pure spin-up state to get a one-particle mixture λ^f. When reading 'Yes' on the upper half of the screen the resulting state has a larger purity than that resulting without reading. Reading the upper left quarter changes the Poincaré vector of the state into $\lambda^f_{x,z}$; similarly it happens that the final state is represented by λ^f_z in case of reading the opposite half of the screen. The right side of the figure shows the state change of the total degenerate one-particle state (1/2) *I*. It does not change during the premeasurement, but it does with the readings corresponding to the subsets of the screen.

The main point is that one can visualize the transformation of the state and the property change. The purity of the resulting state increases when the reading area decreases and vice versa.

4. CHARACTERIZATION OF THE MEASUREMENT

This section reconsiders important notions of the characterization of a quantum mechanical measurement and to see what kinds of notions are still applicable for the joint spin measurement (Busch *et al.*, 1995a, b).

The *probability reproducibility condition* as defined above is satisfied by definition of the observable.

The *calibration condition*

$$tr[E(X)\rho^i_{Ob}] = 1 \Rightarrow tr[P_{x,z}(f^{-1}(X))\rho^f_A]$$

is an always true and therefore empty condition, because it always follows for any nontrivial set X and any initial object state ρ^i_{Ob}: $tr[E(X)\rho^i_{Ob}] < 1$ [for the proof se Kienzler (1996)].

The *value reproducibility condition* states that properties stick to the object, which means $tr[E(Y)\rho^i_{Ob}] = 1 \Rightarrow tr[E(Y)\rho^f_{Ob}] = 1$. Again the premise cannot be reached, but it is possible to state the condition as a maximal condition. That means that eigenstates remain eigenstates. This would be a weaker version of the value reproducibility condition, i.e., $E(Y)\rho^i = \beta\rho^i \Rightarrow E(Y)\rho^f = \beta\rho^f$.

The measurements of the *first kind* fulfil $tr[E(Y)\rho^i_{Ob}] = tr[E(Y)\rho^f_{Ob}]$. Of course this cannot be reached, but the statistical information is saved, i.e., $tr[E(Y)\rho^i_{Ob}] = \frac{1}{2}\, tr[E(Y)\rho^f_{Ob}]$ (for $\mu = 0.606$) for the spin-x and spin-z marginal effects and this means a statistical completeness in x and z.

Repeatability requires a sharp value after the measurement. This cannot be reached for the same reasons as before. The particle cannot have a sharp property before, during, or after the measurement, because the property is intrinsically unsharp.

The presented model shows the need of for very much wider concept of measurement.

5. FURTHER REMARKS

The first remark is on the transition from unsharp to nearly sharp measurements. It is easy to see how the additional homogeneous magnetic field $B_\hbar$ singles out one spin direction $B_0\,(-x, 0, z) \rightarrow B_0\,(-x, 0, z) + (0, 0, B_h)$.

The second remark is on the spin observable as a covariant observable regarding Galilei transformations. The measurement above defines, if the pointer observable is appropriately chosen, a covariant spin observable regarding the rotation R_α in the plane $U(\alpha)\ E(Y)\ U^{-1}(\alpha) = E(R_\alpha(Y))$

Third, a sequential measurement (with reading) converges to a nontrivial state [for more details on these remarks see Kienzler (1996)].

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